Model uncertainty in hedging financial derivatives under constraint

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Abstract

In a discrete-time market, we study model-independent superhedging, while the semi-static superhedging portfolio consists of *three* parts: static positions in liquidly traded vanilla calls, static positions in other tradable, yet possibly less liquid, exotic options, and a dynamic trading strategy in risky assets under certain constraints. By considering the limit order book of each tradable exotic option and employing the Monge-Kantorovich theory of optimal transport, we establish a general superhedging duality, which admits a natural connection to convex risk measures. With the aid of this duality, we derive a model-independent version of the fundamental theorem of asset pricing. The notion "finite optimal arbitrage profit", weaker than no-arbitrage, is also introduced. It is worth noting that our method covers a large class of Delta constraints as well as Gamma constraint.

Setting

We consider a discrete-time market, with a finite horizon $T \in \mathbb{N}$, d risky assets $S = \{S_t\}_{t=0}^T = \{(S_t^1, \cdots, S_t^d)\}_{t=0}^T$, whose initial price $S_0 = x_0 \in \mathbb{R}_+^d$ is given. There is also a risk-free asset $B = \{B_t\}_{t=0}^T$ which is normalized to $B_t \equiv 1$.

Vanilla calls and other tradable options. At time 0, we assume that the vanilla call option with payoff $(S_t^n - K)^+$ can be liquidly traded, at some price $C_n(t, K)$ given in the market, for all $n = 1, \cdots, d$, $t = 1, \cdots, T$, and $K \geq 0$.

The collection of pricing measures consistent with market prices of vanilla calls is therefore

$$\Pi := \left\{ \mathbb{Q} \in \mathcal{P}(\Omega) : \mathbb{E}^{\mathbb{Q}}[(S_t^n - K)^+] = C_n(t, K),\ \forall\, n = 1, \cdots, d,\ t = 1, \cdots, T, \text{ and } K \geq 0 \right\},$$

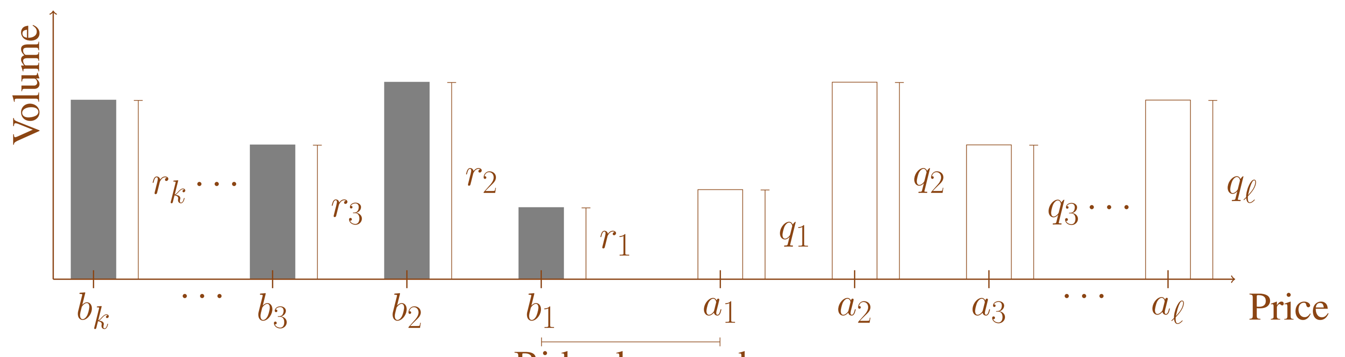
where $\mathcal{P}(\Omega)$ denotes the collection of all probability measures defined on Ω. Under rather standard conditions, the call option prices prescribe fixed marginal probability measures for all pricing measures.

$$\Pi = \{\mathbb{Q} \in \mathcal{P}(\Omega) : \mathbb{Q}_t^n = \mu_t^n,\ \forall\, n = 1, \cdots, d \text{ and } t = 1, \cdots, T\}.$$

Besides vanilla calls, there are other options tradable, while less liquid, at time 0. Let I be a (possibly uncountable) index set. For each $i \in I$, suppose that $\psi_i : \Omega \mapsto \mathbb{R}$ is the payoff function of an option tradable at time 0. Let $\eta \in \mathbb{R}$ be the number of units of ψ_i being traded at time 0, with $\eta \geq 0$ denoting a purchase order and $\eta < 0$ a selling order. Let $c_i(\eta) \in \bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ denote the total cost of trading η units of ψ_i. Throughout this paper, we impose, for each $i \in I$, the following conditions:

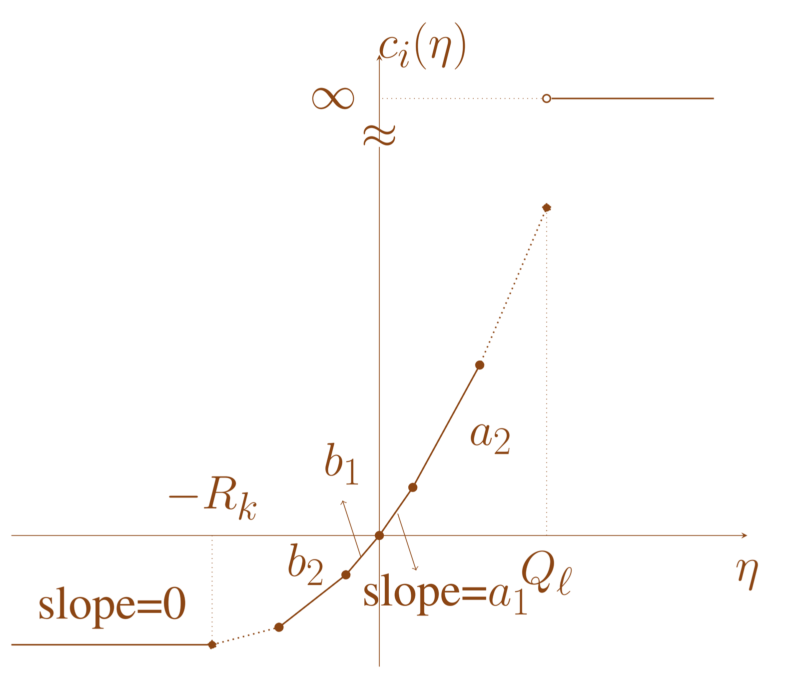
$$c_i : \eta \to c_i(\eta) \text{ is convex and nondecreasing function with } c_i(0) = 0.$$

The rationale behind the properties of pricing function c is in the structure of limit order book (bid-ask chart). Figure below shows the orders at different prices for a risky asset at a particular time. Each bar represents an order; the hight of each bar shows the volume of order and the number below each bar is the price at which the order is made. The gray bars are corresponding to sell (bid) orders, while the white bars are the ask (buy) orders. The second figure shows the price for buying or selling the risky asset at volume η For example if $\eta > q_1$, we can only buy q_1 stocks at price a_1 and the remaining at price a_2 or higher.



Bid-ask spread

In second figure, the pricing function is constructed based on the above bid-ask chart. The slope (derivative) of c_i matches with the prices at the bid-ask chart. Notice that $\eta < 0$ (negative order volume) accounts for selling while $\eta > 0$ accounts for buying.



Definitions

Super-hedging. For a path-dependent exotic option with payoff function $\Phi : (\mathbb{R}_+^d)^T \mapsto \mathbb{R}$, we intend to construct a semi-static superhedging portfolio, which consists of three parts: static positions in vanilla calls, static positions in $\{\psi_i\}_{i \in I}$, and a dynamic trading strategy $\Delta \in \mathcal{S}$. More precisely, consider

$$\mathcal{C} := \left\{ \varphi : \mathbb{R} \mapsto \mathbb{R} : \varphi(x) = a + \sum_{i=1}^n b_i (x - K_i)^+ \text{ for some } a \in \mathbb{R}, n \in \mathbb{N}, b_i \in \mathbb{R} \text{ and } K_i > 0 \right\},$$

$$\mathcal{R}^I := \{\eta = (\eta_i)_{i \in I} \in \mathbb{R} : \eta_i \neq 0 \text{ for finitely many } i\text{'s}\}.$$

$$\Psi_{u,\eta,\Delta}(x) := \sum_{t=1}^T \sum_{n=1}^d u_t^n(x_t^n) + \sum_{i \in I} (\eta_i \psi_i - c_i(\eta_i)) + (\Delta \bullet x)_T \geq \Phi(x) \quad \text{for all } x \in (\mathbb{R}_+^d)^T,$$

$$D(\Phi) := \inf \left\{ \sum_{t=1}^T \sum_{n=1}^d \int_{\mathbb{R}_+} u_t^n d\mu_t^n : u \in \mathcal{U} \text{ satisfies } \exists \eta \in \mathcal{R}^I \text{ and } \Delta \in \mathcal{S} \text{ s.t. } \Psi_{u,\eta,\Delta} \geq \Phi,\ \forall x \in (\mathbb{R}_+^d)^T \right\}.$$

Definition 0.1 (Trading strategies). *We say $\Delta = \{\Delta_t\}_{t=0}^{T-1}$ is a trading strategy if $\Delta_0 \in \mathbb{R}^d$ is a constant and $\Delta_t : (\mathbb{R}_+^d)^t \mapsto \mathbb{R}^d$ is Borel measurable for all $t = 1, \cdots, T-1$. Moreover, the corresponding stochastic integral with respect to $x = (x_1, \cdots, x_T) \in (\mathbb{R}_+^d)^T$ will be expressed as*

$$(\Delta \bullet x)_t := \sum_{i=0}^{t-1} \Delta_i(x_1, \cdots, x_i) \cdot (x_{i+1} - x_i), \text{ for } t = 1, \cdots, T,$$

Also, for any collection $\mathcal{J}$ of trading strategies, we introduce the sub-collections

$$\mathcal{J}^\infty := \{\Delta \in \mathcal{J} : \ \Delta_t : (\mathbb{R}_+^d)^t \mapsto \mathbb{R}^d \text{ is bounded},\ \forall\, t = 1, \cdots, T-1\},$$
$$\mathcal{J}_c^\infty := \{\Delta \in \mathcal{J}^\infty : \ \Delta_t : (\mathbb{R}_+^d)^t \mapsto \mathbb{R}^d \text{ is continuous},\ \forall\, t = 1, \cdots, T-1\}.$$

In this paper, we require the trading strategies to lie in a sub-collection $\mathcal{S}$ of $\mathcal{H}$, prescribed as below.

Definition 0.2 (Adaptively convex portfolio constraint). *$\mathcal{S}$ is a set of trading strategies such that*

(i) $0 \in \mathcal{S}$.

(ii) For any $\Delta, \Delta' \in \mathcal{S}$ and any adapted process h with $h_t \in [0, 1]$ for all $t = 0, \cdots, T-1$,

$$\{h_t \Delta_t + (1 - h_t)\Delta_t'\}_{t=0}^{T-1} \in \mathcal{S}.$$

(iii) For any $\Delta \in \mathcal{S}^\infty$, $\mathbb{Q} \in \Pi$, and $\varepsilon > 0$, there exist a closed set $D_\varepsilon \subseteq (\mathbb{R}_+^d)^T$ and $\Delta^\varepsilon \in \mathcal{S}_c^\infty$ such that

$$\mathbb{Q}(D_\varepsilon) > 1 - \varepsilon \text{ and } \Delta_t = \Delta_t^\varepsilon \text{ on } D_\varepsilon \text{ for } t = 0, \cdots, T-1.$$

Definition 0.3 (Upper variation process). *Given $\mathbb{Q} \in \Pi$, the upper variation process for $\mathcal{S}$ is the increasing process $A^{\mathbb{Q}}$ defined by*

$$A_0^{\mathbb{Q}} := 0, \text{ and } A_{t+1}^{\mathbb{Q}} - A_t^{\mathbb{Q}} := \operatorname*{ess\,sup}_{\Delta \in \mathcal{S}}{}^{\mathbb{Q}} \left\{ \Delta_t \cdot (\mathbb{E}^{\mathbb{Q}}[S_{t+1} \mid \mathcal{F}_t] - S_t) \right\}, \quad t = 0, \cdots, T-1.$$

Definition 0.4 (Measure of misspecification of model).

$$\mathcal{E}_I^{\mathbb{Q}} := \sup_{\eta \in \mathcal{R}^I} \sum_{i \in I} (\eta_i \mathbb{E}^{\mathbb{Q}}[\psi_i] - c_i(\eta_i)) \geq 0 \quad \text{for } \mathbb{Q} \in \Pi.$$

Results

Theorem 0.5. *Suppose that ψ_i is continuous and $|\psi_i|$ have linear growth for all $i \in I$. Then, for any upper semicontinuous function $\Phi : (\mathbb{R}_+^d)^T \mapsto \mathbb{R}$ with linear growth, we have $D(\Phi) = P(\Phi) := \sup_{\mathbb{Q} \in \Pi}\{\mathbb{E}^{\mathbb{Q}}[\Phi - A_T^{\mathbb{Q}}] - \mathcal{E}_I^{\mathbb{Q}}\}$. Moreover, if $\mathcal{Q}_{\mathcal{S},I} \neq \emptyset$, the supremum is attained at some $\mathbb{Q}^* \in \mathcal{Q}_{\mathcal{S},I}$.*

$$\mathcal{P}_{\mathcal{S},I} := \{\mathbb{Q} \in \Pi : \{(\Delta \bullet S)_t\}_{t=0}^T \text{ is a local } \mathbb{Q}\text{-supermartingale, for all } \Delta \in \mathcal{S},$$
$$\text{and } \quad c_i'(0-) \leq \mathbb{E}^{\mathbb{Q}}[\psi_i] \leq c_i'(0+), \text{ for all } i \in I\}$$

Definition 0.6 (Model-independent arbitrage). *We say there is model-independent arbitrage under the constraint $\mathcal{S}$, if there exist $u \in \mathcal{U}_0$, $\eta \in \mathcal{R}^I$, and $\Delta \in \mathcal{S}$ such that*

$$\sum_{t=1}^T \sum_{n=1}^d u_t^n(x_t^n) + \sum_{i \in I} (\eta_i \psi_i(x) - c_i(\eta_i)) + (\Delta \bullet x)_T > 0, \quad \text{for all } x \in (\mathbb{R}_+^d)^T.$$

Theorem 0.7. *Suppose ψ_i is continuous and $|\psi_i|$ have linear growth for all $i \in I$. Then,*

there is no model-independent arbitrage under the constraint $\mathcal{S}$ $\iff$ $\mathcal{P}_{\mathcal{S},I} \neq \emptyset$.

Definition 0.8 (Optimal arbitrage profit). *Consider*

$$G_{\mathcal{S},I} := \sup\{a \in \mathbb{R} : \exists\, u \in \mathcal{U}_0,\ \eta \in \mathcal{R}^I, \text{ and } \Delta \in \mathcal{S} \text{ s.t. } \Psi_{u,\eta,\Delta}(x) > a,\ \forall x \in (\mathbb{R}_+^d)^T\}.$$

By definition, $G_{\mathcal{S},I} \geq 0$. If $G_{\mathcal{S},I} > 0$, we say it is the (model-independent) optimal arbitrage profit.

$$\mathcal{Q}_{\mathcal{S},I} := \{\mathbb{Q} \in Pi : \mathbb{E}^{\mathbb{Q}}[A_T^{\mathbb{Q}}] < \infty, \text{ and } \mathcal{E}_I^{\mathbb{Q}} < \infty\}.$$

Proposition 0.9. (i) $G_{\mathcal{S},I} = 0 \iff \mathcal{P}_{\mathcal{S},I} \neq \emptyset$. $\quad (ii)$ $G_{\mathcal{S},I} < \infty \iff \mathcal{Q}_{\mathcal{S},I} \neq \emptyset$.

Forthcoming Research

Next step is to generalize these results in two directions: (1) Continuous-time, and (2) for more general payoffs than just the upper semi-continuous functions. The former is an ongoing research with a PhD student Langwei Xing at FSU. The later is a joint work yet to be completed with Yu-Jui Huang, Dublin City University.

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