Equilibrium morphologies encountered during erosion, dissolution, and melting

Q: What shape forms when a body melts?

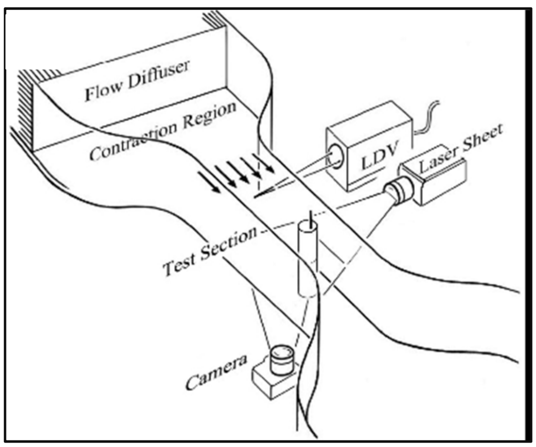
To Mathematicians, this question is known as the Stefan problem. In 1891, Johan Stefan found a class of solutions describing how spheres, cylinders, and planes melt in a self-similar fashion. That is, these bodies shrink down in time while maintaining the same shape, and so they are always similar to an earlier version of themselves. The problem becomes much more complicated, though, when a fluid flow is involved, as is the case when polar ice caps melt, or when landscapes erode.

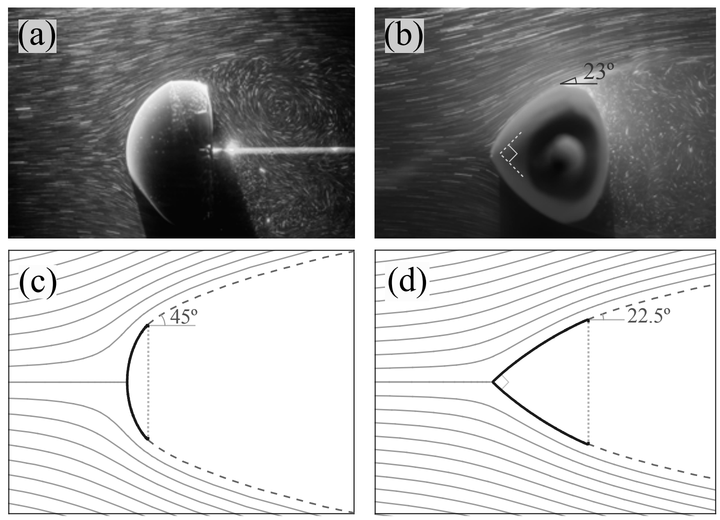
Experiments

Various laboratory experiments have been conducted to determine how fluid flows influence such processes. The experiments typically consist of a water tunnel in which a solid body is inserted (below left). The body can be composed of a material that **melts** (like ice), one that **dissolves** (like solidified sugar), or one that **erodes** (like soft clay).

Interestingly, these processes result in different equilibrium morphologies. As shown below, dissolution produces a rounded front surface (fig a) while erosion sculpts angular features (fig b).

Experiment schematic



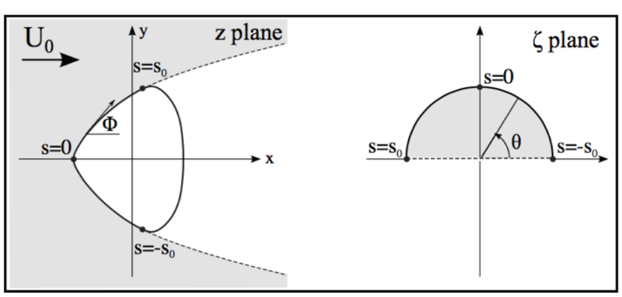


Modeling the flow and shape change

A new mathematical model has given us an understanding of why these different shapes develop. The model uses **complex variables** to describe the surrounding fluid flow. The fluid flow then tells us how the body changes shape. More specifically, the flow is encoded by an **analytical function**,

$$F = \Phi + i\Psi$$

A technique called **conformal mapping** lets us map the original problem, where the body shape might be complicated, to a simpler domain, as shown below.



The Riemann-Hilbert problem

The conformal map gives the following boundary conditions on the imaginary component of F,

$$\boxed{\begin{array}{ll} \Psi = \dfrac{2m}{m+1} \log|\cos|\,\theta & \text{for } \zeta = e^{i\theta} \\ \Psi = 0 & \text{for } \zeta \in (-1,1) \end{array}}$$

Our task is to use these conditions to determine the real component, Φ. This is known as a **Riemann-Hilbert problem**. Then, by unraveling the conformal map, Φ tells us the equilibrium shapes that emerge from each process.

Above, we show the shapes predicted for dissolution (fig c) and erosion (fig d). As you can see, the theory captures the shapes observed in experiments quite well. In particular, the theory shows dissolution to create a rounded surface and erosion to produce points and angles.

The Riemann-Hilbert approach provides a theoretical framework that, in the future, could be used to examine more complicated instances of shape evolution, such as melting ice caps or dissolving karst landscapes. Such research is currently underway.

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